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δN-formalism and superhorizon curvature perturbations

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0. Introduction

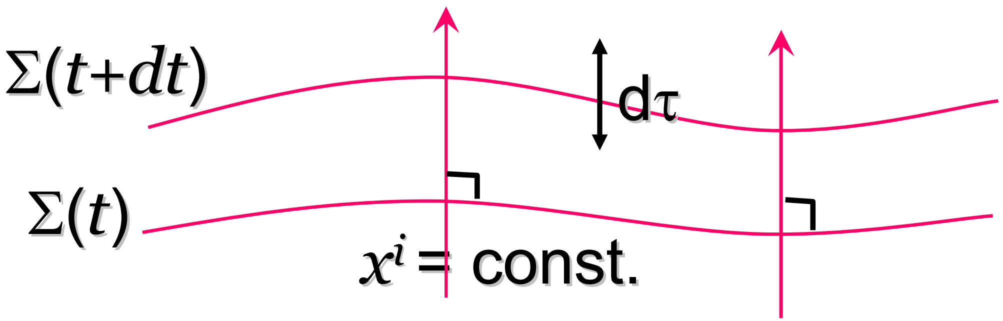
What is δN?

- δN is the perturbation in # of e-folds counted backward in time from a fixed final time t_f
- t_f should be chosen such that the evolution of the universe has become unique by that time.
- δN is equal to conserved NL comoving curvature perturbation on superhorizon scales at $t>t_f$
- δN is essentially independent of gravity theory

1. Linear δN formula

➢ metric (on a spatially flat background)

$$ds^2 = -(1+2A)dt^2 + a^2(t)\left[(1+2\mathcal{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \partial_i \partial_j H_T$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

• propertime along x^i = const.: $d\tau = (1+A)dt$

• curvature perturbation on $\Sigma(t)$: $\mathcal{R} \Leftrightarrow \overset{(3)}{R} = -\dfrac{4}{a^2}\overset{(3)}{\Delta}\mathcal{R}$

• expansion (Hubble parameter): $\tilde{H} = H(1-A) + \partial_t\left[\mathcal{R} + \dfrac{1}{3}\overset{(3)}{\Delta} H_T\right]$

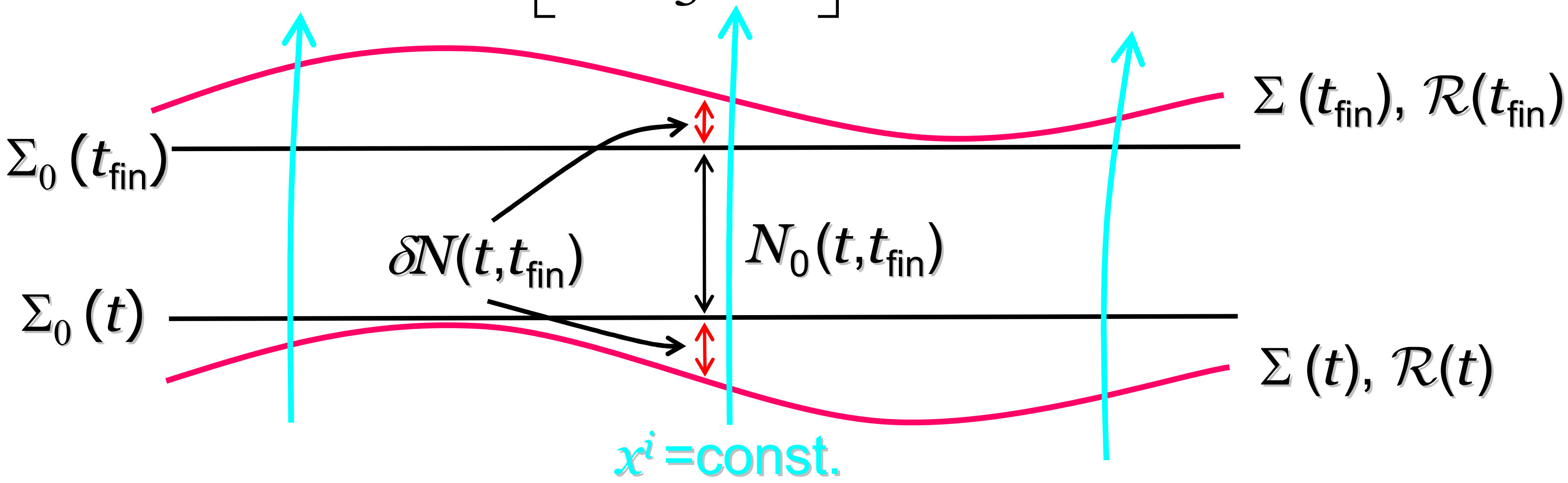
δN formula

MS & Stewart '96

- perturbation in # of e-folds between $\Sigma(t)$ and $\Sigma(t_{\rm fin})$:

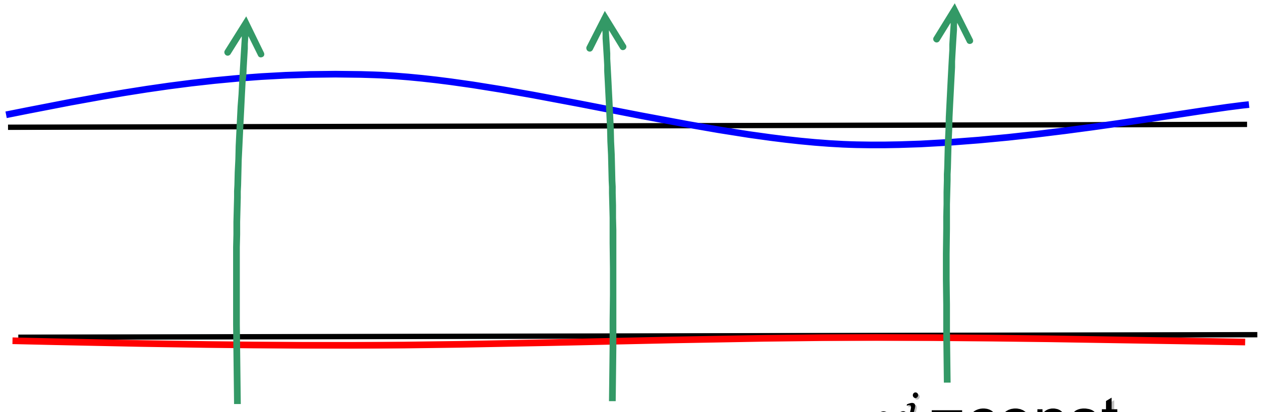
$$\delta N(t;t_{\rm fin}) \equiv \int_t^{t_{\rm fin}} \tilde{H}\, d\tau - \left(\int_t^{t_{\rm fin}} H\, d\tau\right)_{\rm background} \qquad \varepsilon = \frac{\text{horizon size}}{\text{wavelength}}$$

$$= \int_t^{t_{\rm fin}} \partial_t \left[\mathcal{R} + \frac{1}{3}\overset{(3)}{\Delta} E\right] dt = \mathcal{R}(t_{\rm fin}) - \mathcal{R}(t) + O(\varepsilon^2)$$



δN=0 if both $\Sigma(t)$ and $\Sigma(t_{\rm fin})$ are chosen to be 'flat' ($\mathcal{R}$=0).

Choose $\Sigma(t)$ = flat ($\mathcal{R}$=0) and $\Sigma(t_{\rm fin})$ = comoving:



$\Rightarrow$ $$\delta N(t;t_{\rm fin}) = \mathcal{R}_{\rm C}(t_{\rm fin})$$

curvature perturbation on comoving slice
(suffix 'C' for comoving)

The gauge-invariant variable 'ζ' used in the literature
is equal to $\mathcal{R}_{\rm C}$ on superhorizon scales (sometimes $\zeta = -\mathcal{R}_{\rm C}$)

- by definition, $\delta N(t;\, t_{\rm fin})$ is t-independent.
- important to keep in mind that δN is a non-local quantity

2. Non-linear extension

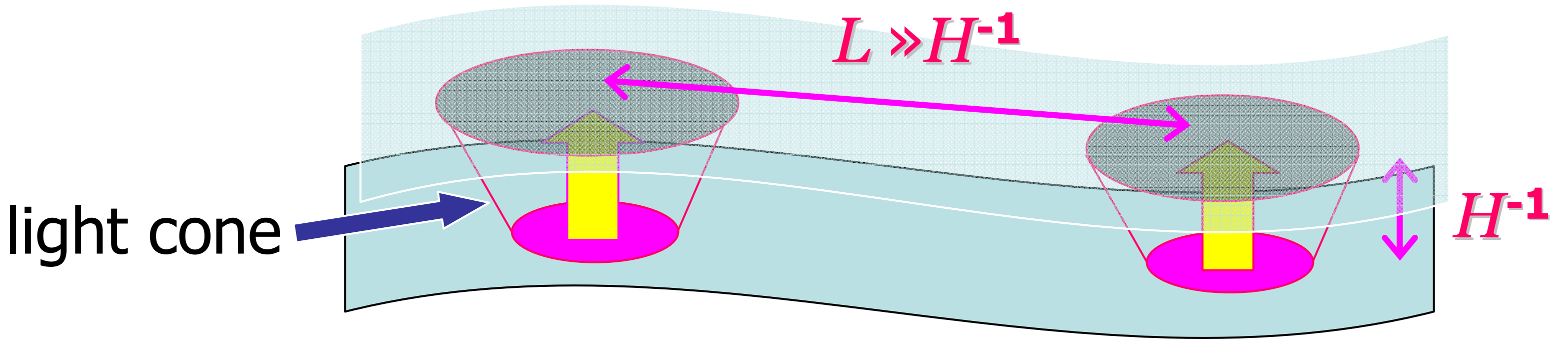
Lyth, Malik & MS ('05)

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i}Q\right| \ll \left|\frac{\partial}{\partial t}Q\right| \sim HQ; \;\; H \sim \sqrt{G\rho}$$

Belinski et al. '70, Tomita '72, Salopek & Bond '90, …

This is a consequence of causality:



- At lowest order, no signal propagates in spatial directions.

Field equations reduce to ODE's

➢ metric on superhorizon scales

- gradient expansion:

$$\partial_i \to \varepsilon \partial_i \,, \quad \varepsilon = \text{expansion parameter}$$

- metric:

$$ds^2 = -\mathcal{N}^2 dt^2 + e^{2\alpha} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

$$\det \tilde{\gamma}_{ij} = 1, \qquad \beta^{\mathrm{i}} = O(\varepsilon)$$

the only non-trivial assumption

contains GW (~ tensor) modes

$$e^{\alpha(t,x^i)} = a(t) e^{\mathcal{R}(t,x^i)} ; \; \mathcal{R} \sim \text{curvature perturbation}$$

e.g., choose $\mathcal{R}(t_*,0) = 0$

fiducial `background’

- Local Friedmann equation & δN formula

$$\tilde{H}^2(t,x^i) = \frac{8\pi G}{3}\rho(t,x^i) + O(\varepsilon^2)$$

$$\tilde{H} \equiv \frac{\partial}{\partial\tau}\alpha = \frac{\partial}{\mathcal{N}\partial t}[\ln a + \mathcal{R}]$$

x^i : comoving (Lagrangean) coordinates.

$d\tau = \mathcal{N}\, dt$: proper time along fluid flow

$$\frac{d}{d\tau}\rho + 3\tilde{H}(\rho + p) = 0 \qquad \boxed{\tilde{H} \equiv -\frac{1}{3(\rho+p)}\frac{\partial}{\partial\tau}\rho}$$

exactly the same as the background equations.

$$N(t_2,t_1) = \int_{t_1}^{t_2} \tilde{H}\, d\tau = N_0(t_2,t_1) + \mathcal{R}(t_2,x^i) - \mathcal{R}(t_1,x^i)$$

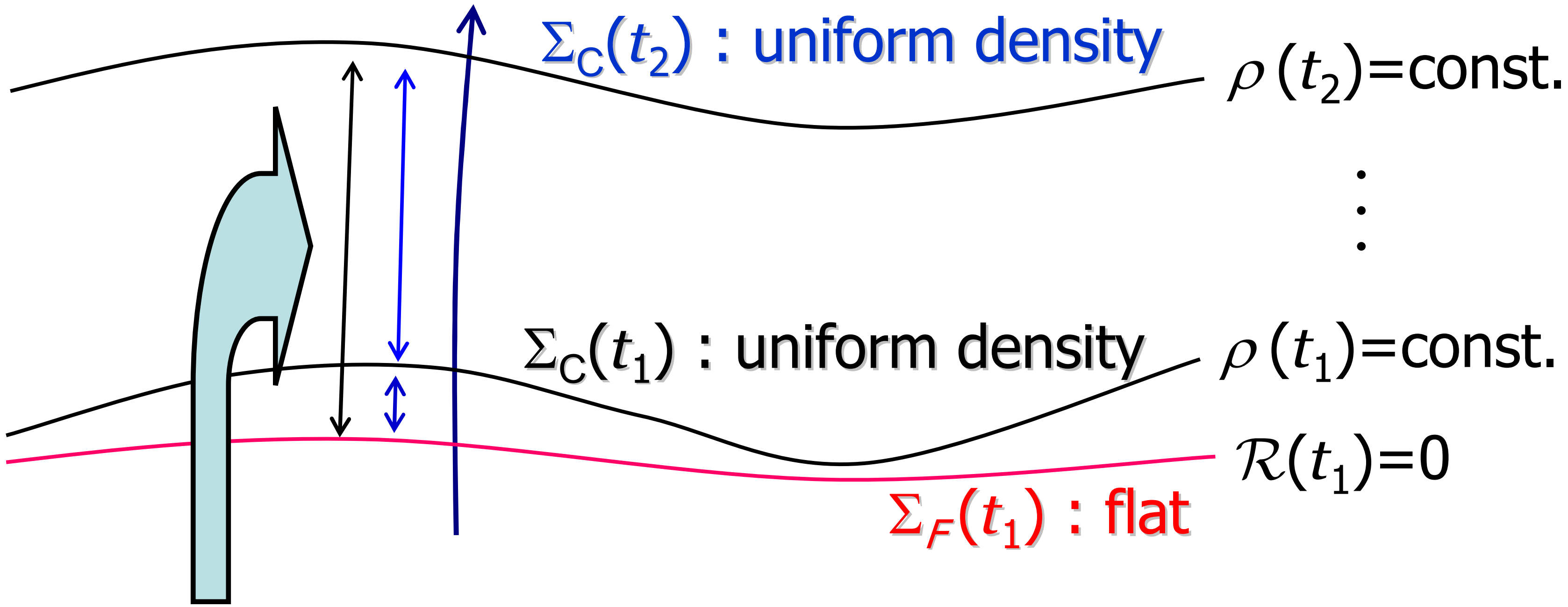
… same as linear theory

$$N_0(t_2,t_1) = \ln\left[a(t_2)/a(t_1)\right] \qquad \boxed{N(t_2,t_1) = -\int_{t_1}^{t_2} dt \frac{1}{3(\rho+p)}\frac{\partial\rho}{\partial t}}$$

➢ δN - formula

Choose slicing such that slice is flat at $t = t_1$ [$\Sigma_F(t_1)$]
and uniform density (=comoving) at $t = t_2$ [$\Sigma_C(t_1)$] :

('flat' slice: $\Sigma(t)$ on which $\mathcal{R} = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$N\left(\Sigma_C(t_2),\Sigma_F(t_1)\right) \equiv N_0\left(t_2,t_1\right) + \delta N_F\left(t_2,t_1;x^i\right)$$

$$= N\left(\Sigma_C(t_2),\Sigma_C(t_1)\right) + N\left(\Sigma_C(t_1),\Sigma_F(t_1)\right)$$

Then

$$\underset{\text{comoving}}{\underset{\parallel}{\mathcal{R}\left(t_2,x^i\right)}}-\overset{=0}{\mathcal{R}\left(t_1,x^i\right)}=\boxed{\mathcal{R}_{\mathrm{C}}\left(t_2,x^i\right)=\delta N_F\left(t_2,t_1;x^i\right)}$$

suffix C for comoving (=uniform ρ)

where $\delta N_F\left(t_2,t_1;x^i\right)+N_0\left(t_2,t_1\right)=-\dfrac{1}{3}\displaystyle\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)}\left.\frac{\partial_t\rho}{\rho+P}\right|_{x^i}dt$

This reduces in linear theory to

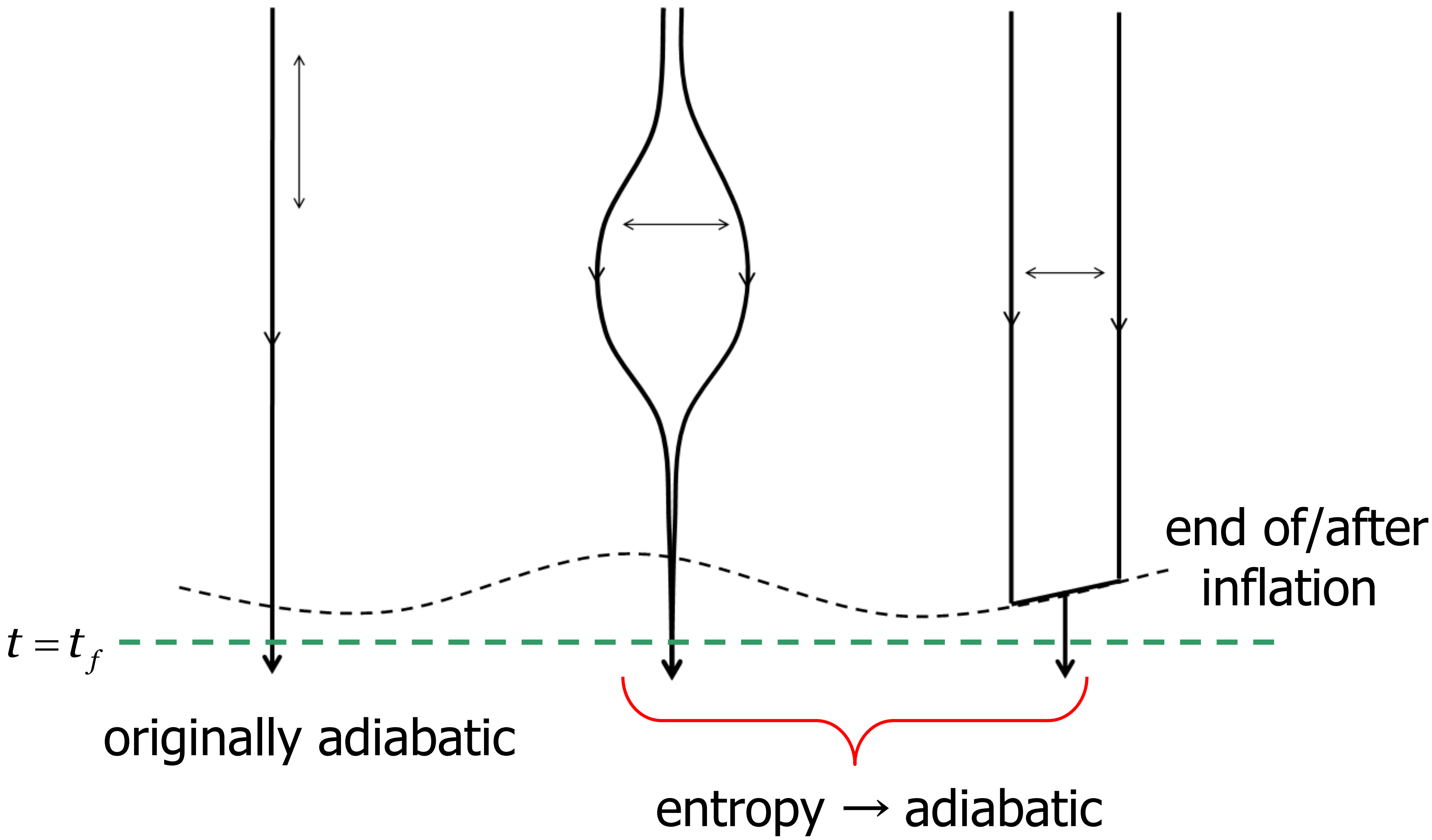
$$\mathcal{R}_{\mathrm{C}}(t_2)=\delta N_F\left(t_1;t_2\right)=\left[\sum_a\frac{\partial N}{\partial\phi^a}\delta\phi_F^a(t_1)\right]$$

N.B. In general $\delta N_F\neq N\left(\Sigma_C(t_1),\Sigma_F(t_1);x^i\right)$

$$\Leftrightarrow\quad N\left(\Sigma_C(t_2),\Sigma_C(t_1);x^i\right)\equiv-\frac{1}{3}\int_{\Sigma_C(t_1)}^{\Sigma_C(t_2)}\left.\frac{\partial_t\rho}{\rho+P}\right|_{x^i}dt\neq N_0\left(t_2,t_1\right)$$

because $\mathcal{R}_{\mathrm{C}}$ is not conserved in general

3 types of δN


end of/after
inflation
$t = t_f$
originally adiabatic
entropy → adiabatic

3. δN for inflation at ‘attractor’ stage

MS & Tanaka (’98), Lyth & Rodriguez (’05),...

- In slow-roll inflation or inflation at an attractor stage, all decaying mode solutions of the (multi-comp) inflaton field ϕ have died out.

- If ϕ is in an attractor regime when the scale of our interest leaves the horizon, $d\phi/dt$ will be a function of ϕ.
  Hence # of e-folds is only a function of ϕ, no matter how complicated the subsequent evolution may be.

$$N = N\left(\phi, \frac{d\phi}{dt}\right), \quad \frac{d\phi}{dt} = f(\phi) \quad \Longrightarrow \quad N = N(\phi)$$

- Nonlinear δN for multi-component inflation :

$$\delta N = N\left(\phi^A + \delta\phi^A\right) - N\left(\phi^A\right)$$
$$= \sum_n \frac{1}{n!} \frac{\partial^n N}{\partial\phi^{A_1}\partial\phi^{A_2}\cdots\partial\phi^{A_n}} \delta\phi^{A_1}\delta\phi^{A_2}\cdots\delta\phi^{A_n}$$

where $\delta\phi = \delta\phi_F$ (on flat slice) at horizon-crossing.

$\delta\phi_F$ may contain non-gaussianity from
subhorizon (quantum) interactions

eg, in DBI inflation

4. Conservation of nonlinear $\mathcal{R}_c$

For adiabatic fluid: $p = p(\rho), \quad \nabla_\alpha T^{\alpha\beta} = 0$

Lyth, Malik & MS ('05)

$$N\left(t_2, t_1; x^i\right) = -\frac{1}{3}\int_{t_1}^{t_2} \frac{\partial_t \rho}{\rho + P(\rho)} dt$$

$$= -\frac{1}{3}\int_{\rho(t_1,x^i)}^{\rho(t_2,x^i)} \frac{d\rho}{\rho + P(\rho)} = \mathcal{R}(t_2, x^i) - \mathcal{R}(t_1, x^i) + \ln\left[\frac{a(t_2)}{a(t_1)}\right]$$

$$\Rightarrow \quad \boxed{\varsigma_{\rm NL}(x^i) \equiv \mathcal{R}(t, x^i) + \frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)} \frac{d\rho}{\rho + P(\rho)}}$$

$\mathcal{R}$ and ρ can be evaluated on any time slice

⋯ slice-independent

r.h.s.=$\mathcal{R}_c$ for ρ =const slice $\Rightarrow$ $\varsigma_{NL} = \mathcal{R}_C$

- applicable to each decoupled fluid

example: $\rho = \sum_a^N \rho_a ; \quad p_a = w_a \rho_a , w_a =$ const.

(N decoupled perfect fluids)

$$\mathcal{R}_a(x^i) = \mathcal{R}_{\text{tot}}(t,x^i) + \int_{\bar{\rho}_a(t)}^{\rho_a(t,x^i)} \frac{d\rho_a}{3(1+w_a)\rho_a}$$

$\mathcal{R}$ on uniform ρ_a

$$= \mathcal{R}_{\text{tot}}(t,x^i) + \frac{1}{3(1+w_a)} \ln \frac{\rho_a(t,x^i)}{\bar{\rho}_a(t)}$$

$$\Rightarrow \quad \rho_a(t,x^i) = \bar{\rho}_a(t) \exp\left[3(1+w_a)\left(\mathcal{R}_a(x^i) - \mathcal{R}_{\text{tot}}(t,x^i)\right)\right]$$

$$\Rightarrow \quad \rho(t) = \sum_a \rho_a(t,x^i)$$

$$= \sum_a \bar{\rho}_a(t) \exp\left[3(1+w_a)\left(\mathcal{R}_a(x^i) - \mathcal{R}_{\text{C,tot}}(t,x^i)\right)\right]$$

uniform density slice

nonlinear version of $\mathcal{R}_{\text{C,tot}}(t,x^i) = \dfrac{\sum_a (1+w_a)\rho_a(t)\mathcal{R}_a(x^i)}{\sum_a (1+w_a)\rho_a(t)}$

Scalar field: P=P(ρ)?

Naruko & MS ('11)

- For a scalar field, this (adiabaticity) condition is non-trivial.

$$S_\phi = \int d^4x\sqrt{-g}\left[W(X,\phi) - G(X,\phi)\Box\phi\right]\ ;\quad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

canonical: $W = X,\ G = 0$

$$\rho = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + V(\phi),\quad p = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 - V(\phi)$$

$d\tau = \mathcal{N}dt,\quad \mathcal{N}:$ lapse

k-essence: $G = 0$

$$\rho = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 W_{,X} - W,\quad p = W(X,\phi)$$

$$X = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2$$

Galileon: $G_{,X} \neq 0$

very complicated...

Field eqn : $\delta S_\phi = 0$

$$-\frac{1}{\sqrt{\gamma}}\partial_\tau\left(\sqrt{\gamma}\Big[(W_{,X} - G_{,\phi})\partial_\tau\phi - G_{,X}(\Box\phi\partial_\tau\phi + \partial_\tau X)\Big]\right) + W_{,\phi} - G_{,\phi}\Box\phi = 0\,,$$

$G = 0$:

Neglecting spatial derivatives, field eqn. becomes

$$\underbrace{\boxed{\left[W_{,X} + W_{,XX}(\partial_\tau\phi)^2\right]\partial_\tau^2\phi + W_{,X\phi}(\partial_\tau\phi)^2 - W_{,\phi}}}_{\equiv\, A(\phi)\partial_\tau\phi} + K\,\underbrace{\boxed{W_{,X}\,\partial_\tau\phi}}_{\equiv\, B(\phi)} = 0$$

$$K = \partial_\tau \ln\sqrt{g^{(3)}} = 3\tilde{H} = 3\partial_\tau(\ln a + \mathcal{R})$$

If in an attractor regime: $\dfrac{\partial\phi}{\partial\tau}\left(=\dfrac{\partial\phi}{\mathcal{N}\partial t}\right) = f(\phi)$

$$A(\phi)\partial_\tau\phi + K\,B(\phi) = 0$$

$\Rightarrow$ $K = -\dfrac{A(\phi)\partial_\tau\phi}{B(\phi)}$ $\Leftrightarrow$ same as $K = -\dfrac{\partial_\tau\rho}{\rho + p(\rho)}$

$$N\left(t_2,t_1;x^i\right)=\frac{1}{3}\int_{t_1}^{t_2}d\tau\,K=N_0\left(t_2,t_1\right)+\mathcal{R}(t_2,x^i)-\mathcal{R}(t_1,x^i)$$

$$=-\int_{t_1}^{t_2}d\tau\frac{\partial_\tau\phi\,A(\phi)}{3B(\phi)}=-\int_{\phi_1}^{\phi_2}d\phi\frac{A(\phi)}{3B(\phi)}\equiv F(\phi_2)-F(\phi_1)$$

- conservation of "slice-independent" NL curvature perturbation

$$\varsigma_{NL}(x^i)=\mathcal{R}(t,x^i)-\left[F\left(\phi(t,x^i)\right)-F\left(\phi_0(t)\right)\right]$$

··· can be evaluated in any slicing

- In particular, for uniform ϕ (=comoving) slicing: $\phi = \phi_0(t)$

$$\varsigma_{NL}(x^i)=\mathcal{R}_c(t,x^i)=\mathcal{R}_c(x^i)$$

valid for "any" theory of gravity

valid for each decoupled scalar field

$G_{,X} \neq 0$:

Field equation

$$\left(W_{,X} - 2G_{,\phi} + 2G_{,X} K \partial_\tau \phi\right)\partial_\tau^2 \phi + G_{,X}(\partial_\tau K + K^2)(\partial_\tau \phi)^2$$
$$+(\partial_\tau G_{,X})K(\partial_\tau \phi)^2 + \left[\partial_\tau (W_{,X} - G_{,\phi}) + K(W_{,X} - 2G_{,\phi})\right]\partial_\tau \phi - W_{,\phi} = 0$$

- appears 2nd time derivative of the metric (~ $\partial_\tau K$),

  → cannot show the conservation w/o using Einstein eqns.

- if we invoke Einstein eqns, we can replace $\partial_\tau K$ with K

⇨ $A(\phi)\partial_\tau \phi + K\, B(\phi) = 0$ at attractor stage

conservation of NL curvature perturbation can be shown only after gravitational equations are used.

ie, valid only if ϕ dominates the universe

8. Summary

- There exists non-linear generalization of δN formula, which may be useful in evaluating non-Gaussianity /nonlinearity from inflation.

- There exists non-linear generalization of conserved $\mathcal{R}_c$ for adiabatic perturbation on superhorizon scales.

  - conservation law applies to each decoupled fluid

  - applies also to each decoupled scalar field provided field eqn contains only 1st time derivative of the metric.

  - for a galieon-type field, which contains 2nd time derivative, the use of gravitational eqn are necessary to show the conservation.